Lecture 27

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1 Orthogonal bases

In this section we will generalize the example from the previous lecture. Let $\{v_1, v_2, \dots, v_n\}$ be an orthogonal basis of the Euclidean space V. Our goal is to find coordinates of the vector u in this basis, i.e such numbers $a_1, a_2, \dots, a_n$, that

$$u = a_1v_1 + a_2v_2 + \cdots + a_nv_n.$$

The familiar way is to write a linear system, and solve it. But since the vectors of the basis are orthogonal, we can do the following. First, let's multiply the expression above by v_1. We'll get:

$$\langle u, v_1\rangle = a_1\langle v_1, v_1\rangle + a_2\langle v_1, v_2\rangle + \cdots + \langle v_1, v_n\rangle.$$

But all products $\langle v_1, v_2\rangle, \dots, \langle v_1, v_n\rangle$ are equal to 0, so we'll have

$$\langle u, v_1\rangle = a_1\langle v_1, v_1\rangle,$$

and thus

$$a_1 = \frac{\langle u, v_1\rangle}{\langle v_1, v_1\rangle}.$$

In the same way multiplying by $v_2, v_3, \dots, v_n$ we will get formulae for other coefficients:

$$a_2 = \frac{\langle u, v_2\rangle}{\langle v_2, v_2\rangle}, \quad \dots, \quad a_n = \frac{\langle u, v_n\rangle}{\langle v_n, v_n\rangle}.$$

Definition 1.1. *The coefficients defined as*

$$a_1 = \frac{\langle u, v_1\rangle}{\langle v_1, v_1\rangle}, \quad \dots, \quad a_n = \frac{\langle u, v_n\rangle}{\langle v_n, v_n\rangle}.$$

*are called **Fourier coefficients** of the vector u with respect to basis $\{v_1, v_2, \dots, v_n\}$.*

Moreover, we proved the following theorem:

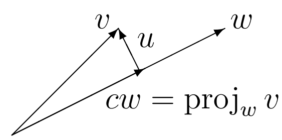
Theorem 1.2. *Let $\{v_1, v_2, \dots, v_n\}$ be an orthogonal basis of the Euclidean space V. Then for any vector u,*

$$u = \frac{\langle u, v_1\rangle}{\langle v_1, v_1\rangle}v_1 + \frac{\langle u, v_2\rangle}{\langle v_2, v_2\rangle}v_2 + \cdots + \frac{\langle u, v_n\rangle}{\langle v_n, v_n\rangle}v_n$$

This expression is called **Fourier decomposition** and can be obtained in any Euclidean space, e.g. the space of continuous functions $C[a, b]$.

2 Projections

In this lecture we will continue study orthogonality. We'll start now with the projection of a vector to another vector.



The projection of the vector v along the vector w is the vector $\operatorname{proj}_w v = cw$ proportional to w, such that $u = v - cw$ is orthogonal to w. So, to find projection, we have to determine the number c, and then we can simply multiply it by vector w. After that we will be able to find the perpendicular from v onto w, i.e. u.

Since we know that u is orthogonal to w, then we can write

$$\langle u, w\rangle = 0.$$

But

$$u = v - cw,$$

so

$$\langle v - cw, w\rangle = 0 \quad \Leftrightarrow \quad \langle v, w\rangle - c\langle w, w\rangle = 0.$$

From the last equality we can find c:

$$c = \frac{\langle v, w\rangle}{\langle w, w\rangle}$$

So, the projection of the vector v along the vector w is given by the following formula:

$$\operatorname{proj}_w v = \frac{\langle v, w\rangle}{\langle w, w\rangle} w$$

The orthogonal component u is equal to

$$u = v - \operatorname{proj}_w v = v - \frac{\langle v, w\rangle}{\langle w, w\rangle} w$$

The length of this perpendicular u will be the distance between the point, corresponding to vector v and the line, which goes through 0 with direction vector w.

Example 2.1. *Let' s find the distance from the point* $(1, 3)$ *to the line* $y = x$. *The direction vector of this line is* $(1, 1)$. *So, in our terms we have the following data:*

$$v = (1, 3), \quad w = (1, 1).$$

Let's compute projection of v *along* w*:*

$$\operatorname{proj}_w v = \frac{\langle v, w\rangle}{\langle w, w\rangle} w = \frac{1\cdot 1 + 3\cdot 1}{1\cdot 1 + 1\cdot 1} w = \frac{4}{2} w = 2w = 2(1, 1) = (2, 2).$$

Now, orthogonal component is

$$u = v - \text{proj}_w v = (1,3) - (2,2) = (-1,1).$$

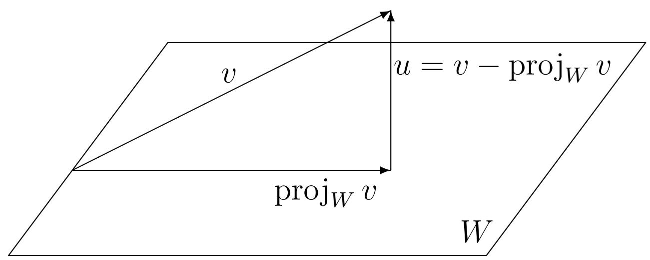
The distance d between the point and the line is equal to the length of the perpendicular, i.e.

$$d = \|u\| = \sqrt{1+1} = \sqrt{2}.$$

So, needed distance is equal to $\sqrt{2}$.

This method gives us a way to find a distance between the line through the origin and the point.

But we may want to consider more difficult problem of finding the distance between the point and the plane, or a subspace of any other dimension!



We will generalize our constructions. Let we have a subspace (i.e., plane) W, and we have its orthogonal basis $\{w_1, w_2, \dots, w_n\}$.

Theorem 2.2. *The projection $\text{proj}_W v$ of any vector v along W is the following vector:*

$$\text{proj}_W v = \frac{\langle v, w_1\rangle}{\langle w_1, w_1\rangle} w_1 + \frac{\langle v, w_2\rangle}{\langle w_2, w_2\rangle} w_2 + \cdots + \frac{\langle v, w_n\rangle}{\langle w_n, w_n\rangle} w_n$$

In particular, it means, that

$$u = v - \text{proj}_W v = v - \left(\frac{\langle v, w_1\rangle}{\langle w_1, w_1\rangle} w_1 + \frac{\langle v, w_2\rangle}{\langle w_2, w_2\rangle} w_2 + \cdots + \frac{\langle v, w_n\rangle}{\langle w_n, w_n\rangle} w_n \right)$$

is orthogonal to the subspace W.

Proof. To prove it, we will multiply u by any vector w_i. We'll have:

$$\langle u, w_i\rangle = \langle v, w_i\rangle - \left(\frac{\langle v, w_1\rangle}{\langle w_1, w_1\rangle} \langle w_1, w_i\rangle + \frac{\langle v, w_2\rangle}{\langle w_2, w_2\rangle} \langle w_2, w_i\rangle + \cdots + \frac{\langle v, w_n\rangle}{\langle w_n, w_n\rangle} \langle w_n, w_i\rangle \right)$$

All products $\langle w_j, w_i\rangle$ are equal to 0 except $\langle w_i, w_i\rangle$. So, we have:

$$\begin{aligned} \langle u, w_i\rangle &= \langle v, w_i\rangle - \frac{\langle v, w_i\rangle}{\langle w_i, w_i\rangle} \langle w_i, w_i\rangle \\ &= \langle v, w_i\rangle - \langle v, w_i\rangle \\ &= 0. \end{aligned}$$

So, u is orthogonal to every w_i, and thus it is orthogonal to W. □

So, if we have a subspace with the orthogonal basis in it, and a vector, we can compute a distance between them. But often it happens that the basis in the subspace is not orthogonal, so our next goal will be to develop algorithm of finding orthogonal bases.

3 Gram-Schmidt orthogonalization process

Let we have any basis $\{v_1, v_2, \dots, v_n\}$ in the Euclidean space. We want to construct orthogonal basis $\{w_1, w_2, \dots, w_n\}$ of this space. We will do it as follows.

$$\begin{aligned}
&w_1 = v_1 \\
&w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 \\
&w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 \\
&\dots \\
&w_n = v_n - \frac{\langle v_n, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_n, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \frac{\langle v_n, w_3\rangle}{\langle w_3, w_3\rangle} w_3 - \dots - \frac{\langle v_n, w_{n-1}\rangle}{\langle w_{n-1}, w_{n-1}\rangle} w_{n-1}
\end{aligned}$$

Actually, each time we're subtracting the projection to the space, spanned by the vectors, already orthogonalized.

After this algorithm we will have orthogonal basis $\}w_1, w_2, \dots, w_n\}$.

Example 3.1. *Let*

$$\begin{aligned}
v_1 &= (1, 1, -1, -2); \\
v_2 &= (5, 8, -2, -3); \\
v_3 &= (3, 9, 3, 8).
\end{aligned}$$

Let's apply the Gram-Schmidt orthogonalization process to these vectors.

$$w_1 = v_1 = (1, 1, -1, -2).$$

Now, let's find w_2:

$$\begin{aligned}
w_2 &= v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 \\
&= (5, 8, -2, -3) - \frac{5 \cdot 1 + 8 \cdot 1 + (-2) \cdot (-1) + (-3) \cdot (-2)}{1 \cdot 1 + 1 \cdot 1 + (-1) \cdot (-1) + (-2) \cdot (-2)} (1, 1, -1, -2) \\
&= (5, 8, -2, -3) - \frac{21}{7} (1, 1, -1, -2) \\
&= (5, 8, -2, -3) - (3, 3, -3, -6) \\
&= (2, 5, 1, 3).
\end{aligned}$$

Now, we can find w_3:

$$\begin{aligned}
w_3 &= v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 \\
&= (3,9,3,8) - \frac{6+45+3+24}{4+25+1+9}(2,5,1,3) - \frac{3+9-3-16}{1+1+1+4}(1,1,-1,-2) \\
&= (3,9,3,8) - \frac{78}{39}(2,5,1,3) - \frac{-7}{7}(1,1,-1,-2) \\
&= (3,9,3,8) - 2(2,5,1,3) + (1,1,-1,-2) \\
&= (0,0,0,0).
\end{aligned}$$

Finally, we got:

$$\begin{aligned}
w_1 &= (1,1,-1,-2); \\
w_2 &= (2,5,1,3); \\
w_3 &= (0,0,0,0).
\end{aligned}$$

The third vector is a zero-vector, so we don't need it. Actually, it means that vectors v_1, v_2 and v_3 are in the same plane, so, the basis of this plane consists of 2 vectors, and the orthogonal basis consists of w_1 and w_2.

Again, this process is very general, and can be used in any Euclidean space, i.e. the space of continuous functions $C[a,b]$.

4 Distance between a vector and a subspace

Now when we know how to find orthogonal bases of the subspace, we can find distances between the vector (or a point, corresponding to this vector) and a subspace, for example a plane which goes through origin.

Let we want to find a distance between vector v and a subspace with any basis. Then we should first orthogonalize the basis of the subspace using Gram-Schmidt orthogonalization process, and then compute projections of v along vectors of basis. Then, subtracting projections from v we will get a vector, which is orthogonal to the subspace. Its length will be equal to the needed distance.

Example 4.1. *Let we have a plane P in the 3-dimensional space with the following basis: $v_1 = (1,0,-1)$ and $v_2 = (-1,1,0)$. Let's find the distance between point $(1,2,3)$ and this plane.*

First we should orthogonalize the basis of the plane.

$$\begin{aligned}
w_1 &= v_1 = (1, 0, -1) \\
w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\
&= (-1, 1, 0) - \frac{-1}{1+1}(1, 0, -1) \\
&= (-1, 1, 0) + \frac{1}{2}(1, 0, -1) \\
&= (-\frac{1}{2}, 1, -\frac{1}{2}).
\end{aligned}$$

Now we should find projection of v along this plane.

$$\begin{aligned}
\operatorname{proj}_P v &= \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 \\
&= \frac{1-3}{1+1}(1, 0, -1) + \frac{-\frac{1}{2} + 2 - \frac{3}{2}}{\frac{1}{4} + 1 + \frac{1}{4}}(-\frac{1}{2}, 1, -\frac{1}{2}) \\
&= (-1, 0, 1).
\end{aligned}$$

The vector, orthogonal to this plane from the point $(1, 2, 3)$ is

$$u = v - \operatorname{proj}_P v = (1, 2, 3) - (-1, 0, 1) = (2, 2, 2).$$

So, the distance is

$$d = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}.$$